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Massless spin-$\frac{5}{2}$ wave equations

R K Loide

Tallinn Polytechnic Institute, 200 026 Tallinn, USSR

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Abstract. The general form of the massless spin-$\frac{5}{2}$ wave equation for a symmetrical tensor-bispinor is given. It is shown that there exists a whole class of massless equations. The general form of gauge transformation and source constraint is given, and the invariant scalar product and Lagrangian are also presented. The γ-tracelessness of a gauge field is not needed. It appears that the Lagrangian does not determine the equation uniquely; the knowledge of the invariant scalar product is also needed.

1. Introduction

The development of supergravity (van Nieuwenhuizen 1981) points out the importance of high-spin massless particles in particle physics. Supersymmetry provides two possible fermionic partners to the free graviton: spin-$\frac{3}{2}$ or spin-$\frac{5}{2}$ massless fields can accompany it in an irreducible supermultiplet. In the spin-$\frac{3}{2}$ gravitino case supergravity provides the consistent coupling of spin-$\frac{3}{2}$ and spin-2 fields, but if there are no particles of spin higher than two it is not possible to accommodate the known particle spectrum and symmetries. For that reason it is of interest to analyse massless spin-$\frac{5}{2}$ fields and consistency conditions more thoroughly.

In this paper we re-examine the description of massless spin-$\frac{5}{2}$ particles with the help of the symmetric tensor-bispinor $\psi^{\mu\nu}$. The action for $\psi^{\mu\nu}$ was given in Schwinger (1970), and Fang and Fronsdal (1978) generalised the results to higher-spin fermions, exploiting the massive equations of Singh and Hagen (1974). The detailed derivation and analysis of massless spin-$\frac{5}{2}$ equations was given in Berends *et al* (1979a, b, 1980). When gravity is coupled to spin-$\frac{5}{2}$ the theory becomes inconsistent (Aragone and Deser 1979, 1980, Berends *et al* 1980) due to the γ-tracelessness of the matter gauge parameter. Therefore it seems that there is no hope of obtaining a consistent spin-$\frac{5}{2}$ field theory. Using the modified version of the spin-projection technique (Loide 1984) we demonstrate that the symmetrical tensor-bispinor admits a whole class of massless wave equations. The general forms of the gauge transformations and source constraints are given. It appears that the main reason for the inconsistency, the γ-tracelessness of the gauge parameter, is not needed.

We use the general formalism of spin-projection operators previously exploited in the massive spin-$\frac{3}{2}$ case in Loide (1984). Our formalism differs from that of Berends *et al* (1979b) and Berends and van Reisen (1980). Our projection operators are connected with a fixed representation of the Lorentz group and have fixed non-localities which depend on the representations used. In the construction of equations the projection operators β^s_{ij} ('roots' in the terminology of the root method) are needed. The operation with operators $\Pi^s_{ij}\partial$ instead of β^s_{ij} as in Berends *et al* (1979b) is quite troublesome. The massless equations can be derived similarly to the massive ones. From the set of all massive equations we obtain the subset of equations which in the $m=0$ case admit gauge transformations and give massless equations. The gauge transformations and source constraints are also expressed with the help of spin-projection operators. The invariant scalar product and Lagrangian are given. It appears that in the massless case the knowledge of the Lagrangian is not sufficient to derive an equation because the explicit form of the corresponding equation depends on the choice of scalar product.

In this paper only the free field equations are given. We do not consider the coupling with other fields. The paper is planned as follows. Section 2 gives the derivation of the massless equations for $\psi^{\mu\nu}$ in the formalism of spin-projection operators. Section 3 gives the covariant form, and § 4 presents the invariant scalar product and Lagrangian. Section 5 gives some examples.

2. General formalism of spin-projection operators

The symmetrical tensor-bispinor $\psi^{\mu\nu}_{\alpha}$ transforms according to the representation $(\frac{3}{2},1)\oplus(1,\frac{1}{2})\oplus(\frac{1}{2},0)\oplus(0,\frac{1}{2})\oplus(\frac{1}{2},1)\oplus(1,\frac{3}{2})$ of the Lorentz group. We decompose $\psi^{\mu\nu}$ into the direct sum $\psi=\psi_1\oplus\psi_2\oplus\psi_3$, where ψ_1 transforms according to the representation $1=(\frac{3}{2},1)\oplus(1,\frac{3}{2})$, ψ_2 according to $2=(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ and ψ_3 according to $3=(\frac{1}{2},0)\oplus(0,\frac{1}{2})$.

Using the spin-projection operators β^{s}_{ij} (see appendix), similarly to Loide (1984) we obtain the following general first-order equation:

$$\mathrm{i}\sqrt{\Box}\begin{vmatrix} \beta^{5/2}_{11}+\frac{2}{3}\beta^{3/2}_{11}+\frac{1}{3}\beta^{1/2}_{11} & a(\beta^{3/2}_{12}+2\sqrt{\frac{2}{5}}\beta^{1/2}_{12}) & 0 \\ b(\beta^{3/2}_{21}+2\sqrt{\frac{2}{5}}\beta^{1/2}_{21}) & c(\beta^{3/2}_{22}+\frac{1}{2}\beta^{1/2}_{22}) & d\beta^{1/2}_{23} \\ 0 & e\beta^{1/2}_{32} & f\beta^{1/2}_{33} \end{vmatrix}\begin{vmatrix}\psi_1\\ \psi_2\\ \psi_3\end{vmatrix}=0, \tag{2.1}$$

where a, b, c, d, e and f are free parameters.

If we write (2.1) as $\pi\psi=0$ it is useful to decompose $\pi=\pi^{5/2}+\pi^{3/2}+\pi^{1/2}$, where

$$\pi^{5/2}=\mathrm{i}\sqrt{\Box}\begin{vmatrix}\beta^{5/2}_{11} & 0 & 0\\ 0&0&0\\ 0&0&0\end{vmatrix},\qquad \pi^{3/2}=\mathrm{i}\sqrt{\Box}\begin{vmatrix}\frac{2}{3}\beta^{3/2}_{11} & a\beta^{3/2}_{12} & 0\\ b\beta^{3/2}_{21} & c\beta^{3/2}_{22} & 0\\ 0&0&0\end{vmatrix},$$

$$\pi^{1/2}=\mathrm{i}\sqrt{\Box}\begin{vmatrix}\frac{1}{3}\beta^{1/2}_{11} & a2\sqrt{\frac{2}{5}}\beta^{1/2}_{12} & 0\\ b2\sqrt{\frac{2}{5}}\beta^{1/2}_{21} & \frac{1}{2}c\beta^{1/2}_{22} & d\beta^{1/2}_{23}\\ 0 & e\beta^{1/2}_{32} & f\beta^{1/2}_{33}\end{vmatrix}. \tag{2.2}$$

The corresponding reduced matrices π_s (Gel'fand–Yaglom spin blocks) are the following:

$$\pi_{5/2}=1,\qquad \pi_{3/2}=\begin{vmatrix}\frac{2}{3} & a\\ b & c\end{vmatrix},$$
$$\pi_{1/2}=\begin{vmatrix}\frac{1}{3} & a2\sqrt{\frac{2}{5}} & 0\\ b2\sqrt{\frac{2}{5}} & c/2 & d\\ 0 & e & f\end{vmatrix}. \tag{2.3}$$

If we consider the massive equations $\pi\psi=m\psi$ it is easy to verify that these are all multi-mass equations. In order to have a single spin-$\frac{5}{2}$ equation the matrices $\pi_{3/2}$ and $\pi_{1/2}$ must both be nilpotent which is impossible.

Next we consider gauge transformations and demonstrate that for the gauge invariance of equation (2.1) it is necessary and sufficient that $\det \pi_{3/2} = \det \pi_{1/2} = 0$. The last conditions explain the difficulties in the zero mass limit of the spin-$\frac{5}{2}$ and spin-3 equations treated in Berends *et al* (1979b) and Berends and van Reisen (1980). In Berends *et al* (1979b) it is shown that for the description of a single massive spin-$\frac{5}{2}$ particle one must use two representations, the symmetrical tensor-bispinor $\psi^{\mu\nu}$ and bispinor λ, whereas in the massless case only $\psi^{\mu\nu}$ is needed. The reason lies in the fact that in the massive case one must demand the reduced matrices $\pi_{3/2}$ and $\pi_{1/2}$ to be nilpotent. The conditions $\det \pi_{3/2} = \det \pi_{1/2} = 0$, needed in the massless case, are weaker and need fewer representations to satisfy them. The same considerations are also valid in the spin-3 case. Also it is obvious why we cannot use the traceless tensor-bispinor $\psi^{\mu\nu}(\psi^{\mu}{}_{\mu} = 0)$, since there are not enough free parameters to fulfil the conditions $\det \pi_{3/2} = \det \pi_{1/2} = 0$. So the symmetrical tensor-bispinor representation offers a minimal dimensional spin-$\frac{5}{2}$ massless theory that can be derived from the Lagrangian.

For the gauge field we choose the vector-bispinor ε^{μ}. If we demand, as usual, that the gauge transformations must be linear in derivatives, the gauge field must transform according to the representation $4 = (1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$, because it is the only representation which is linked with all three representations ψ_1, ψ_2 and ψ_3, and is consistent with a given equation. We shall denote the gauge field ψ_4 and not ψ_2 because ψ_4 is extracted from a vector-bispinor ε^{μ}. Now the most general gauge transformation is written in the form

$$\begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{vmatrix} \to \begin{vmatrix} \psi_1 + \sqrt{\Box}(\beta_{14}^{3/2} + 2\sqrt{\frac{2}{5}}\beta_{14}^{1/2})\psi_4 \\ \psi_2 + \alpha\sqrt{\Box}(\beta_{24}^{3/2} + \frac{1}{2}\beta_{24}^{1/2})\psi_4 \\ \psi_3 + \beta\sqrt{\Box}\beta_{34}^{1/2}\psi_4 \end{vmatrix} \tag{2.4}$$

where α and β are some parameters.

Demanding the gauge invariance of equation (2.1), and using (A1), in the spin-$\frac{3}{2}$ case we get the equation

$$\begin{vmatrix} \frac{2}{3} & a \\ b & c \end{vmatrix} \begin{vmatrix} 1 \\ \alpha \end{vmatrix} = 0$$

which has non-trivial solutions iff $\det \pi_{3/2} = 0$, i.e. if

$$ab = 2c/3. \tag{2.5}$$

Now α is expressed with the help of the coefficient a,

$$\alpha = -2/3a. \tag{2.6}$$

Similar considerations in the spin-$\frac{1}{2}$ case give $\det \pi_{1/2} = 0$, which from (2.5) leads to

$$ed = -27cf/10. \tag{2.7}$$

The coefficient β in (2.4) has the form

$$\beta = e/3af. \tag{2.8}$$

In conclusion: equation (2.1) is invariant under the gauge transformation (2.4) if the coefficients a, b, c, d, e and f satisfy (2.5) and (2.7). The parameters α and β in the gauge transformation (2.4) are expressed from (2.6) and (2.8). The conditions (2.5) and (2.7) followed from $\det \pi_{3/2} = \det \pi_{1/2} = 0$.

Analogously it is possible to verify that the gauge field must indeed transform according to the representation 4. If we add to ψ_4 the bispinor in order to use the full vector-bispinor ε^{μ} we obtain that equation (2.1) admits gauge transformations when $a=0$ which is not consistent with our previous conditions. The latter in turn means that there are no massless equations having the gauge transformation $\psi^{\mu\nu} \rightarrow \psi^{\mu\nu} + \partial^{\mu}\varepsilon^{\nu} + \partial^{\nu}\varepsilon^{\mu}$.

When the conditions (2.5) and (2.7) are valid there exists an operator Q^z with the property $Q^z\pi = 0$. Q^z is expressed as

$$Q^z = \sqrt{\Box}\,|\beta_{41}^{3/2} + 2\sqrt{\tfrac{2}{5}}\beta_{41}^{1/2}\ \gamma(\beta_{42}^{3/2} + \tfrac{1}{2}\beta_{42}^{1/2})\ \delta\beta_{43}^{1/2}|. \tag{2.9}$$

Using the properties of the operators β_{ij}^{s} (equation (A1)), we obtain

$$\gamma = -2/3b, \qquad \delta = d/3bf. \tag{2.10}$$

Operator Q^z gives the source constraint. In the case of an external source J we have an equation $\pi\psi = J$ and because $Q^z\pi = 0$, J must satisfy $Q^zJ = 0$.

3. The general covariant form

The results of § 2 may be written in a convenient covariant form exploiting the expressions of the spin-projection operators β_{ij}^{s} ((A2) and (A4)).

The general covariant form of (2.1) is

$$\begin{aligned} &\mathrm{i}\partial\!\!\!/\psi^{\mu\nu} + \mathrm{i}A(\partial^{\mu}\gamma_{\kappa}\psi^{\kappa\nu} + \partial^{\nu}\gamma_{\kappa}\psi^{\kappa\mu}) + \mathrm{i}B(\gamma^{\mu}\partial_{\kappa}\psi^{\kappa\nu} + \gamma^{\nu}\partial_{\kappa}\psi^{\kappa\mu}) \\ &\quad + \mathrm{i}C(\gamma^{\mu}\partial\!\!\!/\gamma_{\kappa}\psi^{\kappa\nu} + \gamma^{\nu}\partial\!\!\!/\gamma_{\kappa}\psi^{\kappa\mu}) + \mathrm{i}D(\partial^{\mu}\gamma^{\nu} + \partial^{\nu}\gamma^{\mu})\psi^{\kappa}{}_{\kappa} \\ &\quad + \mathrm{i}E\eta^{\mu\nu}(\partial_{\kappa}\gamma_{\lambda} + \partial_{\lambda}\gamma_{\kappa})\psi^{\kappa\lambda} + \mathrm{i}F\eta^{\mu\nu}\partial\!\!\!/\psi^{\kappa}{}_{\kappa} = 0, \end{aligned} \tag{3.1}$$

where we have denoted

$$\begin{aligned} &A = a/\sqrt{5} - \tfrac{1}{3}, && B = b/\sqrt{5} - \tfrac{1}{3}, \\ &C = \frac{c}{6} - \frac{a+b}{6\sqrt{5}} + \frac{2}{9}, && D = \frac{1}{18} + \frac{c}{12} - \frac{a+b}{6\sqrt{5}} - \frac{d}{6}, \\ &E = \frac{1}{18} - \frac{c}{12} - \frac{a+b}{6\sqrt{5}} - \frac{e}{6}, && F = \frac{c}{8} + \frac{f}{4} + \frac{e+d}{12} - \frac{1}{6}. \end{aligned} \tag{3.2}$$

Equation (3.1) is invariant under the gauge transformation

$$\begin{aligned} \psi^{\mu\nu} \rightarrow{} & \psi^{\mu\nu} + \partial^{\mu}\varepsilon^{\nu} + \partial^{\nu}\varepsilon^{\mu} + \tfrac{1}{6}(\sqrt{5}\alpha - 1)(\gamma^{\mu}\partial\!\!\!/\varepsilon^{\nu} + \gamma^{\nu}\partial\!\!\!/\varepsilon^{\mu}) \\ & - \tfrac{1}{6}(\sqrt{5}\alpha/2 + 1)(\partial^{\mu}\gamma^{\nu} + \partial^{\nu}\gamma^{\mu})\gamma_{\kappa}\varepsilon^{\kappa} + \tfrac{1}{4}\sqrt{5}(\tfrac{1}{2}\alpha + \tfrac{1}{3}\beta)\eta^{\mu\nu}\partial\!\!\!/\gamma_{\kappa}\varepsilon^{\kappa} \\ & - \tfrac{1}{3}(1 + \tfrac{1}{2}\sqrt{5}\alpha + \sqrt{5}\beta)\eta^{\mu\nu}\partial_{\kappa}\varepsilon^{\kappa} \end{aligned} \tag{3.3}$$

where α and β are expressed from (2.6) and (2.8), and the conditions (2.5) and (2.7) are satisfied.

The general source constraint $Q^zJ = 0$ has the form

$$\begin{aligned} &2\partial_{\rho}J^{\rho\kappa} + \tfrac{1}{3}(\sqrt{5}\gamma - 1)\partial\!\!\!/\gamma_{\rho}J^{\rho\kappa} - \tfrac{1}{3}(\tfrac{1}{2}\sqrt{5}\gamma + 1)\gamma^{\kappa}\partial_{\rho}\gamma_{\sigma}J^{\rho\sigma} \\ &\quad + \tfrac{1}{4}\sqrt{5}(\tfrac{1}{2}\gamma + \tfrac{1}{3}\delta)\gamma^{\kappa}\partial\!\!\!/J^{\rho}{}_{\rho} - \tfrac{1}{3}(1 + \tfrac{1}{2}\sqrt{5}\gamma + \sqrt{5}\delta)\partial^{\kappa}J^{\rho}{}_{\rho} = 0, \end{aligned} \tag{3.4}$$

where γ and δ are expressed from (2.10).

As we have mentioned in § 2, the gauge transformation extracts from ε^μ the representation $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$. For that reason it is not possible to choose the gauge $\gamma_\mu\psi^{\mu\nu}=0$. The gauge must be chosen so that it corresponds to the representation $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$: $\gamma_\mu\psi^{\mu\nu}-\frac{1}{4}\gamma^\nu\psi^\mu{}_\mu=0$. The last gauge follows from the equation

$$\sqrt{5}\alpha(\not{\partial}\varepsilon^\nu-\tfrac{1}{2}\partial^\nu\gamma_\kappa\varepsilon^\kappa-\tfrac{1}{2}\gamma^\nu\partial_\kappa\varepsilon^\kappa+\tfrac{3}{8}\gamma^\nu\not{\partial}\gamma_\kappa\varepsilon^\kappa)+\gamma_\mu\psi^{\mu\nu}-\tfrac{1}{4}\gamma^\nu\psi^\kappa{}_\kappa=0,$$

which is (due to the results of Loide (1984)) always solvable.

4. Invariant scalar product, Lagrangian

The invariant scalar product which is consistent with our equation (3.1) is

$$\tilde{\psi}_{\mu\nu}\psi^{\mu\nu}=\bar{\psi}_{\mu\nu}\psi^{\mu\nu}-\tfrac{1}{3}(1-a/b)\bar{\psi}_{\mu\rho}\gamma^\rho\gamma_\sigma\psi^{\sigma\mu}-\tfrac{1}{12}(2+a/b-3ad/be)\bar{\psi}^\mu{}_\mu\psi^\nu{}_\nu. \tag{4.1}$$

This scalar product defines a conjugated wavefunction $\tilde{\psi}_{\mu\nu}$

$$\tilde{\psi}_{\mu\nu}=\bar{\psi}_{\rho\sigma}[\eta^\rho{}_\mu\eta^\sigma{}_\nu-\tfrac{1}{3}(1-a/b)\eta^\rho{}_\nu\gamma^\sigma\gamma_\mu-\tfrac{1}{12}(2+a/b-3ad/be)\eta^{\rho\sigma}\eta_{\mu\nu}]. \tag{4.2}$$

The symmetrical choice of coefficients ($a=b$, $d=e$) gives the simplest scalar product $\tilde{\psi}_{\mu\nu}\psi^{\mu\nu}=\bar{\psi}_{\mu\nu}\psi^{\mu\nu}$ and only in that case is the conjugated wavefunction $\tilde{\psi}_{\mu\nu}$ equal to the Dirac conjugated wavefunction $\bar{\psi}_{\mu\nu}$.

The Lagrangian is obtained from an equation $\pi\psi=0$ in the following way:

$$L=\tilde{\psi}\pi\psi. \tag{4.3}$$

Using (4.2) and (3.1) we obtain

$$\begin{aligned}L={}&\mathrm{i}\bar{\psi}_{\mu\nu}\not{\partial}\psi^{\mu\nu}+\mathrm{i}\tfrac{2}{15}\sqrt{5}(3a-\sqrt{5})(\bar{\psi}_{\mu\nu}\partial^\nu\gamma_\kappa\psi^{\kappa\mu}+\bar{\psi}_{\mu\nu}\gamma^\nu\partial_\kappa\psi^{\kappa\mu})\\&+\mathrm{i}\tfrac{1}{90}(45a^2-12\sqrt{5}a+40)\bar{\psi}_{\mu\nu}\gamma^\nu\not{\partial}\gamma_\kappa\psi^{\kappa\mu}\\&+\mathrm{i}(\tfrac{1}{9}-2\sqrt{5}a/15-ac/6b-ad/3b)(\bar{\psi}_{\mu\nu}\partial^\mu\gamma^\nu\psi^\kappa{}_\kappa\\&+\bar{\psi}^\mu{}_\mu\partial_\kappa\gamma_\lambda\psi^{\kappa\lambda})+\mathrm{i}(ac/8b+ad/6b+adf/4be-\tfrac{1}{6})\bar{\psi}^\mu{}_\mu\not{\partial}\psi^\nu{}_\nu.\end{aligned} \tag{4.4}$$

The Lagrangian (4.4) is invariant under the following transformation of coefficients:

$$a\to a,\qquad b\to\kappa b,\qquad c\to\kappa c,\qquad d\to\kappa d,\qquad e\to\lambda e,\qquad f\to\lambda f, \tag{4.5}$$

where κ and λ are arbitrary non-zero real parameters. The transformation (4.5) preserves the conditions $\det\pi_{3/2}=\det\pi_{1/2}=0$ and gives a subset of massless equations which correspond to the same gauge transformation. The given equations have different source constraints, different scalar products and the corresponding massive equations have a different mass spectrum. So we see that in the massless case the knowledge of the Lagrangian is not sufficient to derive an equation uniquely. In addition to L the invariant scalar product or conjugated wavefunction is needed.

The transformation (4.5) is equivalent to the following redefinition of equation (3.1). If equation (3.1) is denoted by $S^{\mu\nu}=0$ then the transformation (4.5) leads to the equation $S^{\mu\nu}+\rho(\gamma^\mu\gamma_\kappa S^{\kappa\nu}+\gamma^\nu\gamma_\kappa S^{\kappa\mu})+\sigma\eta^{\mu\nu}S^\kappa{}_\kappa=0$ with parameters ρ and σ which depend on the choice of κ and λ.

In § 5 examples 2 and 3 are related with the help of transformation (4.5) where $\kappa=\lambda=-\frac{1}{2}$.

There also exists the transformation of coefficients which preserves the conditions $\det \pi_{3/2} = \det \pi_{1/2} = 0$ and extracts a subset of equations which correspond to the same source constraint. The corresponding transformation is the following:

$$b \to b, \qquad a \to \kappa a, \qquad c \to \kappa c, \qquad e \to \kappa e, \qquad d \to \lambda d, \qquad f \to \lambda f, \tag{4.6}$$

where κ and λ are arbitrary non-zero real parameters. The given equations have different scalar products, different Lagrangians and the corresponding massive equations have a different mass spectrum.

The transformation (4.6) is equivalent to the following field redefinition (Berends *et al* 1979b, Berends and van Reisen 1980):

$$\psi^{\mu\nu} \to \psi^{\mu\nu} + \rho(\gamma^\mu \gamma_\kappa \psi^{\kappa\nu} + \gamma^\nu \gamma_\kappa \psi^{\kappa\mu}) + \sigma \eta^{\mu\nu} \psi^\kappa{}_\kappa$$

where ρ and σ are some parameters.

Examples 1 and 3 in § 5 are related with the help of transformation (4.6), where $\kappa = \lambda = -\frac{1}{2}$.

Concluding this section it should be mentioned that the gauge transformation (3.3) $\psi \to \psi + Q_g \varepsilon$ and source constraint $Q^z J = 0$ are related through the invariant scalar product (4.1). If, for example, we consider the interaction term $\tilde{J}_{\mu\nu}\psi^{\mu\nu}$ then in gauge transformation we obtain an additional term $\tilde{J}_{\mu\nu} Q^{\mu\nu}_{g\kappa} \varepsilon^\kappa$ which after an integration leads to $(Q^{z\kappa}{}_{\mu\nu} J^{\mu\nu})\varepsilon_\kappa$. The last term guarantees that the Lagrangian is invariant under gauge transformations.

5. Examples

In the previous sections we have shown that there exists a whole set of massless spin-$\frac{5}{2}$ equations. Since the general expressions are quite complicated we give here some examples by a particular choice of coefficients a, b, c, d, e and f.

1. $A = -1$, $B = C = D = F = 0$ $(a = -\frac{2}{3}\sqrt{5},\ b = \frac{1}{3}\sqrt{5},\ c = \frac{5}{3},\ e = d = \frac{3}{2},\ f = \frac{1}{2})$

$$\mathrm{i}\partial\!\!\!/ \psi^{\mu\nu} - \mathrm{i}\partial^\mu \gamma_\kappa \psi^{\kappa\nu} - \mathrm{i}\partial^\nu \gamma_\kappa \psi^{\kappa\mu} = 0. \tag{5.1}$$

In the massive case $\pi\psi = m\psi$ we have a single mass equation which describes one spin-$\frac{5}{2}$, one spin-$\frac{3}{2}$ and two spin-$\frac{1}{2}$ particles with the same mass m.

Equation (5.1) is invariant under the gauge transformation $(\alpha = 1/\sqrt{5},\ \beta = -3/2\sqrt{5})$

$$\psi^{\mu\nu} \to \psi^{\mu\nu} + \partial^\mu \varepsilon^\nu + \partial^\nu \varepsilon^\mu - \tfrac{1}{4}(\partial^\mu \gamma^\nu + \partial^\nu \gamma^\mu)\gamma_\kappa \varepsilon^\kappa. \tag{5.2}$$

The source constraint takes the form $(\gamma = -2/\sqrt{5},\ \delta = 3/\sqrt{5})$

$$2\partial_\rho J^{\rho\kappa} - \partial\!\!\!/ \gamma_\rho J^{\rho\kappa} - \partial^\kappa J^\rho{}_\rho = 0. \tag{5.3}$$

Due to $a \neq b$ the scalar product takes the form

$$\tilde{\psi}_{\mu\nu}\psi^{\mu\nu} = \bar{\psi}_{\mu\nu}\psi^{\mu\nu} - \bar{\psi}_{\mu\rho}\gamma^\rho \gamma_\sigma \psi^{\sigma\mu} - \tfrac{1}{2}\bar{\psi}^\mu{}_\mu \psi^\nu{}_\nu \tag{5.4}$$

and the Lagrangian is

$$\begin{aligned} L = {} & \mathrm{i}\bar{\psi}_{\mu\nu}\partial\!\!\!/ \psi^{\mu\nu} - 2\mathrm{i}\bar{\psi}_{\mu\nu}\partial^\nu \gamma_\kappa \psi^{\kappa\mu} - 2\mathrm{i}\bar{\psi}_{\mu\nu}\gamma^\nu \partial_\kappa \psi^{\kappa\mu} + 2\mathrm{i}\bar{\psi}_{\mu\nu}\gamma^\nu \partial\!\!\!/ \gamma_\kappa \psi^{\kappa\mu} \\ & + \mathrm{i}\bar{\psi}_{\mu\nu}\partial^\mu \gamma^\nu \psi^\kappa{}_\kappa + \mathrm{i}\bar{\psi}^\rho{}_\rho \partial_\kappa \gamma_\lambda \psi^{\kappa\lambda} - \tfrac{1}{2}\mathrm{i}\bar{\psi}^\mu{}_\mu \partial\!\!\!/ \psi^\nu{}_\nu. \end{aligned} \tag{5.5}$$

Equation (5.1) was derived and analysed in Berends *et al* (1979a, b, 1980), but the general expression of the gauge transformation (5.2) was not used. In the above cited papers where spin-$\frac{5}{2}$ was investigated, the gauge transformation was written in the form $\psi^{\mu\nu} \to \psi^{\mu\nu} + \partial^\mu \varepsilon^\nu + \partial^\nu \varepsilon^\mu$, where ε^μ must satisfy $\gamma_\mu \varepsilon^\mu = 0$. In the free field case the given transformation is equivalent to our transformation (5.2), but in the presence of interactions the γ-tracelessness of the gauge parameter ε^μ may lead to inconsistencies.

2. $B=-1$, $A=C=D=E=F=0$ ($a=\frac{1}{3}\sqrt{5}$, $b=-\frac{2}{3}\sqrt{5}$, $c=-\frac{5}{3}$, $e=d=\frac{3}{2}$, $f=\frac{1}{2}$)

$$\mathrm{i}\partial\!\!\!/\psi^{\mu\nu} - \mathrm{i}\gamma^\mu \partial_\kappa \psi^{\kappa\nu} - \mathrm{i}\gamma^\nu \partial_\kappa \psi^{\kappa\mu} = 0. \tag{5.6}$$

In the massive case we have the same mass spectrum as in the previous case. Equation (5.6) is invariant under the gauge transformation ($\alpha = -2/\sqrt{5}$, $\beta = 3/\sqrt{5}$)

$$\psi^{\mu\nu} \to \psi^{\mu\nu} + \partial^\mu \varepsilon^\nu + \partial^\nu \varepsilon^\mu - \tfrac{1}{2}(\gamma^\mu \partial\!\!\!/ \varepsilon^\nu + \gamma^\nu \partial\!\!\!/ \varepsilon^\mu) - \eta^{\mu\nu}\partial_\kappa \varepsilon^\kappa. \tag{5.7}$$

The source constraint is ($\gamma = 1/\sqrt{5}$, $\delta = -3/2\sqrt{5}$)

$$2\partial_\rho J^{\rho\kappa} - \tfrac{1}{2}\gamma^\kappa \partial_\rho \gamma_\sigma J^{\rho\sigma} = 0. \tag{5.8}$$

The scalar product is

$$\tilde{\psi}_{\mu\nu}\psi^{\mu\nu} = \bar{\psi}_{\mu\nu}\psi^{\mu\nu} - \tfrac{1}{2}\bar{\psi}_{\mu\rho}\gamma^\rho\gamma_\sigma\psi^{\sigma\mu} - \tfrac{1}{4}\bar{\psi}^\mu{}_\mu \psi^\nu{}_\nu \tag{5.9}$$

and the Lagrangian takes the form

$$L = \mathrm{i}\bar{\psi}_{\mu\nu}\partial\!\!\!/\psi^{\mu\nu} + \tfrac{1}{2}\mathrm{i}\bar{\psi}_{\mu\nu}\gamma^\nu \partial\!\!\!/ \gamma_\kappa \psi^{\kappa\mu} - \tfrac{1}{4}\mathrm{i}\bar{\psi}^\mu{}_\mu \partial\!\!\!/ \psi^\nu{}_\nu. \tag{5.10}$$

From (5.7) one can see that it is possible to choose the gauge $\gamma_\mu\psi^{\mu\nu} - \frac{1}{4}\gamma^\nu\psi^\mu{}_\mu = 0$, but now $\gamma_\mu \varepsilon^\mu = 0$ is not needed.

3. $A=B=D=E=0$, $C=-F=\frac{1}{4}$ ($a=b=\frac{1}{3}\sqrt{5}$, $c=\frac{5}{6}$, $e=d=-\frac{3}{4}$, $f=-\frac{1}{4}$)

$$\mathrm{i}\partial\!\!\!/\psi^{\mu\nu} + \tfrac{1}{4}\mathrm{i}(\gamma^\mu \partial\!\!\!/ \gamma_\kappa \psi^{\kappa\nu} + \gamma^\nu \partial\!\!\!/ \gamma_\kappa \psi^{\kappa\mu}) - \tfrac{1}{4}\mathrm{i}\eta^{\mu\nu}\partial\!\!\!/\psi^\kappa{}_\kappa = 0. \tag{5.11}$$

In the massive case we have one spin-$\frac{5}{2}$ particle with mass m, one spin-$\frac{3}{2}$ particle with mass $2m/3$ and two spin-$\frac{1}{2}$ particles with masses m and $2m/3$.

Equation (5.11) is invariant under the same gauge transformation (5.7) as equation (5.6) and has source constraint (5.2) as equation (5.1).

Now the scalar product is $\tilde{\psi}_{\mu\nu}\psi^{\mu\nu} = \bar{\psi}_{\mu\nu}\psi^{\mu\nu}$ and the Lagrangian is the same as in the previous case, (5.10). Here we have an example which confirms that in the massless case the knowledge of the Lagrangian is not sufficient to derive an equation uniquely. In addition to L the invariant scalar product or conjugated wavefunction is needed. Equation (5.11) follows from (5.10) by varying it with respect to $\tilde{\psi}_{\mu\nu} = \bar{\psi}_{\mu\nu}$, equation (5.6) by varying (5.10) with respect to $\tilde{\psi}_{\mu\nu} = \bar{\psi}_{\rho\sigma}(\eta^\rho{}_\mu \eta^\sigma{}_\nu - \frac{1}{2}\eta^\rho{}_\nu \gamma^\sigma \gamma_\mu - \frac{1}{4}\eta^{\rho\sigma}\eta_{\mu\nu})$.

4. Fields $\psi^{\mu\nu}$ and λ

In this paper we do not consider the other realisations of massless spin-$\frac{5}{2}$ equations. As an example we write down here one possible spin-$\frac{5}{2}$ equation using the symmetrical tensor-bispinor $\psi^{\mu\nu}$ and bispinor λ. We have used the multi-mass equation and contrary to the results of Berends *et al* (1979b) the bispinor field λ does not decouple.

$$\begin{aligned} &\mathrm{i}\partial\!\!\!/\psi^{\mu\nu} + \tfrac{1}{4}\mathrm{i}(\gamma^\mu \partial\!\!\!/ \gamma_\kappa \psi^{\kappa\nu} + \gamma^\nu \partial\!\!\!/ \gamma_\kappa \psi^{\kappa\mu}) - \tfrac{1}{4}\mathrm{i}\eta^{\mu\nu}\partial\!\!\!/\psi^\kappa{}_\kappa \\ &\qquad - \tfrac{1}{4}\mathrm{i}\eta^{\mu\nu}\partial\!\!\!/\lambda + \tfrac{1}{4}\mathrm{i}(\partial^\mu\gamma^\nu + \partial^\nu\gamma^\mu)\lambda = 0, \\ &\qquad -\tfrac{1}{4}\mathrm{i}\partial\!\!\!/\psi^\kappa{}_\kappa + \tfrac{1}{4}\mathrm{i}(\partial_\kappa\gamma_\lambda + \partial_\lambda\gamma_\kappa)\psi^{\kappa\lambda} - \tfrac{1}{4}\mathrm{i}\partial\!\!\!/\lambda = 0. \end{aligned} \tag{5.12}$$

Equation (5.12) is invariant under the gauge transformation

$$\psi^{\mu\nu} \to \psi^{\mu\nu} + \partial^{\mu}\varepsilon^{\nu} + \partial^{\nu}\varepsilon^{\mu} - \tfrac{1}{2}(\gamma^{\mu}\not\partial\varepsilon^{\nu} + \gamma^{\nu}\not\partial\varepsilon^{\mu}) - \eta^{\mu\nu}\partial_{\kappa}\varepsilon^{\kappa} + \tfrac{1}{4}\eta^{\mu\nu}\not\partial\gamma_{\kappa}\varepsilon^{\kappa},$$
$$\lambda \to \lambda - \tfrac{1}{2}\not\partial\gamma_{\kappa}\varepsilon^{\kappa}. \tag{5.13}$$

This equation is derived from the Lagrangian

$$L = \mathrm{i}\bar{\psi}_{\mu\nu}\not\partial\psi^{\mu\nu} + \tfrac{1}{2}\mathrm{i}\bar{\psi}_{\mu\nu}\gamma^{\nu}\not\partial\gamma_{\kappa}\psi^{\kappa\mu} - \tfrac{1}{4}\mathrm{i}\bar{\psi}^{\mu}{}_{\mu}\not\partial\psi^{\nu}{}_{\nu} - \tfrac{1}{4}\mathrm{i}\bar{\psi}^{\mu}{}_{\mu}\not\partial\lambda$$
$$+ \tfrac{1}{2}\mathrm{i}\bar{\psi}_{\mu\nu}\partial^{\mu}\gamma^{\nu}\lambda - \tfrac{1}{4}\mathrm{i}\bar{\lambda}\not\partial\psi^{\nu}{}_{\nu} + \tfrac{1}{2}\mathrm{i}\bar{\lambda}\partial_{\kappa}\gamma_{\lambda}\psi^{\kappa\lambda} - \tfrac{1}{4}\mathrm{i}\bar{\lambda}\not\partial\lambda \tag{5.14}$$

by varying with respect to $\bar{\psi}_{\mu\nu}$ and $\bar{\lambda}$. The invariant scalar product is $\bar{\psi}_{\mu\nu}\psi^{\mu\nu} + \bar{\lambda}\lambda$.

6. Conclusions

In this paper we have proposed a class of massless spin-$\frac{5}{2}$ wave equations for a symmetrical tensor-bispinor $\psi^{\mu\nu}$. The general form of gauge transformation and source constraint is given. These equations are derivable from the Lagrangian by varying with respect to a conjugated wavefunction $\tilde{\psi}_{\mu\nu}$.

All massless equations are equivalent in the free field case, but may lead to different results when coupled to other fields. The massive equations corresponding to the massless ones have in general a different mass spectrum.

As we have shown the massless equations can be derived from the massive ones. One must write down the most general massive equation for a given representation and find a subset of equations which in the $m=0$ case admit gauge invariance. In Berends and van Reisen (1980) the problem of massless limit of massive amplitude is analysed. As we have shown in the massless spin-$\frac{3}{2}$ case (Loide and Polt 1985), the massless limit of massive amplitude between two external sources does not in general exist. In the cases when the massless limit exists this procedure is unphysical because it differs from the massless amplitude and for that reason does not lead to massless propagators. The same considerations are also valid in our spin-$\frac{5}{2}$ case and give the same results. Comparing our results with the results of Fronsdal (1980a, b) on the smooth massless limit, it should be mentioned that this limit does not in general exist and the conditions on sources given in Fronsdal (1980a, b) are too restrictive.

Appendix. Spin-projection operators

The symmetrical tensor-bispinor $\psi_{\mu\nu}$ is presented as a direct sum $\psi = \psi_1 \oplus \psi_2 \oplus \psi_3$, where ψ_1 transforms according to the representation $1 = (\frac{3}{2}, 1) \oplus (1, \frac{3}{2})$, ψ_2 according to the representation $2 = (1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ and ψ_3 according to the representation $3 = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

We use the projection operators Π^{s}_{ij} and β^{s}_{ij} $(i, j = 1, 2, 3)$ which satisfy the relations

$$\Pi^{s}_{ij}\Pi^{s'}_{kl} = \delta_{ss'}\delta_{jk}\Pi^{s}_{il},$$
$$\Pi^{s}_{ij}\beta^{s'}_{kl} = \beta^{s}_{ij}\Pi^{s'}_{kl} = \delta_{ss'}\delta_{jk}\beta^{s}_{il}, \tag{A1}$$
$$\beta^{s}_{ij}\beta^{s'}_{kl} = \delta_{ss'}\delta_{jk}\Pi^{s}_{il}.$$

The operators Π^{s}_{ij} connect the representations which are not linked. $\Pi^{5/2}_{11}$, for example, is $\Pi^{5/2}_{11} = P^{5/2}_{(3/2,1)(3/2,1)} \oplus P^{5/2}_{(1,3/2)(1,3/2)}$ where the projection operator $P^{5/2}_{(3/2,1)(3/2,1)}$ extracts from ψ_1 spin-$\frac{5}{2}$ and the result transforms according to the representation $(\frac{3}{2}, 1)$. Operators β^s_{ij} connect the representations which are linked. $\beta^{5/2}_{11}$, for example has the form $\beta^{5/2}_{11} = P^{5/2}_{(3/2,1)(1,3/2)} \oplus P^{5/2}_{(1,3/2)(3/2,1)}$. Since Π^s_{ij} are generated from β^s_{ij} we give only the operators β^s_{ij}.

$$
\begin{aligned}
\sqrt{\Box}\,\beta^{5/2}_{11} &= \tfrac{1}{2}\underline{1} - \tfrac{1}{5}\underline{2} + \tfrac{1}{10}\underline{3} - \tfrac{1}{10}(\underline{4}+\underline{5}) + \tfrac{1}{5}(\underline{8}+\underline{9}) - \tfrac{2}{5}\underline{10} - \tfrac{1}{10}\underline{11} + \tfrac{1}{5}(\underline{12}+\underline{13}) + \tfrac{2}{5}\underline{14}\\
\sqrt{\Box}\,\beta^{3/2}_{11} &= \tfrac{1}{45}\underline{2} + \tfrac{1}{60}\underline{3} - \tfrac{1}{10}(\underline{4}+\underline{5}) + \tfrac{1}{18}(\underline{6}+\underline{7}) - \tfrac{2}{15}(\underline{8}+\underline{9}) + \tfrac{3}{5}\underline{10} + \tfrac{11}{90}\underline{11} - \tfrac{2}{15}(\underline{12}+\underline{13}) - \tfrac{8}{5}\underline{14},\\
\sqrt{\Box}\,\beta^{3/2}_{22} &= \tfrac{1}{9}\underline{2} + \tfrac{1}{12}\underline{3} - \tfrac{1}{18}(\underline{6}+\underline{7}) - \tfrac{1}{18}\underline{11},\\
\sqrt{\Box}\,\beta^{3/2}_{12} &= (1/3\sqrt{5})(-\tfrac{1}{3}\underline{2} - \tfrac{1}{4}\underline{3} + \tfrac{3}{2}\underline{4} - \tfrac{5}{6}\underline{6} + \tfrac{1}{6}\underline{7} + 2\,\underline{8} + \tfrac{2}{3}\underline{11} - 4\,\underline{12}),\\
\sqrt{\Box}\,\beta^{3/2}_{21} &= (1/3\sqrt{5})(-\tfrac{1}{3}\underline{2} - \tfrac{1}{4}\underline{3} + \tfrac{3}{2}\underline{5} + \tfrac{1}{6}\underline{6} - \tfrac{5}{6}\underline{7} + 2\,\underline{9} + \tfrac{2}{3}\underline{11} - 4\,\underline{13}),\\
\sqrt{\Box}\,\beta^{1/2}_{11} &= \tfrac{1}{18}\underline{2} + \tfrac{1}{18}(\underline{6}+\underline{7}) - \tfrac{1}{3}(\underline{8}+\underline{9}) + \tfrac{1}{18}\underline{11} - \tfrac{1}{3}(\underline{12}+\underline{13}) + 2\,\underline{14},\\
\sqrt{\Box}\,\beta^{1/2}_{22} &= \tfrac{1}{36}\underline{2} - \tfrac{1}{18}(\underline{6}+\underline{7}) + \tfrac{1}{9}\underline{11},\\
\sqrt{\Box}\,\beta^{1/2}_{33} &= \tfrac{1}{4}\underline{2},\\
\sqrt{\Box}\,\beta^{1/2}_{12} &= (1/3\sqrt{2})(\tfrac{1}{6}\underline{2} + \tfrac{1}{6}\underline{6} - \tfrac{1}{3}\underline{7} - \underline{8} - \tfrac{1}{3}\underline{11} + 2\,\underline{12}),\\
\sqrt{\Box}\,\beta^{1/2}_{21} &= (1/3\sqrt{2})(\tfrac{1}{6}\underline{2} - \tfrac{1}{3}\underline{6} + \tfrac{1}{6}\underline{7} - \underline{9} - \tfrac{1}{3}\underline{11} + 2\,\underline{13}),\\
\sqrt{\Box}\,\beta^{1/2}_{23} &= \tfrac{1}{12}\underline{2} - \tfrac{1}{6}\underline{6},\\
\sqrt{\Box}\,\beta^{1/2}_{32} &= \tfrac{1}{12}\underline{2} - \tfrac{1}{6}7,
\end{aligned}
\tag{A2}
$$

where we have denoted

$$
\begin{aligned}
&(\underline{1})^{\mu\nu}{}_{\kappa\lambda} = \eta^{\mu}{}_{\kappa}\eta^{\nu}{}_{\lambda} + \eta^{\mu}{}_{\lambda}\eta^{\nu}{}_{\kappa}, \qquad (\underline{2})^{\mu\nu}{}_{\kappa\lambda} = \eta^{\mu\nu}\not\partial\,\eta_{\kappa\lambda},\\
&(\underline{3})^{\mu\nu}{}_{\kappa\lambda} = \eta^{\mu}{}_{\kappa}\gamma^{\nu}\not\partial\,\gamma_{\lambda} + \eta^{\mu}{}_{\lambda}\gamma^{\nu}\not\partial\,\gamma_{\kappa} + \eta^{\nu}{}_{\kappa}\gamma^{\mu}\not\partial\,\gamma_{\lambda} + \eta^{\nu}{}_{\lambda}\gamma^{\mu}\not\partial\,\gamma_{\kappa},\\
&(\underline{4})^{\mu\nu}{}_{\kappa\lambda} = \eta^{\mu}{}_{\kappa}\partial^{\nu}\gamma_{\lambda} + \eta^{\mu}{}_{\lambda}\partial^{\nu}\gamma_{\kappa} + \eta^{\nu}{}_{\kappa}\partial^{\mu}\gamma_{\lambda} + \eta^{\nu}{}_{\lambda}\partial^{\mu}\gamma_{\kappa},\\
&(\underline{5})^{\mu\nu}{}_{\kappa\lambda} = \eta^{\mu}{}_{\kappa}\gamma^{\nu}\partial_{\lambda} + \eta^{\mu}{}_{\lambda}\gamma^{\nu}\partial_{\kappa} + \eta^{\nu}{}_{\kappa}\gamma^{\mu}\partial_{\lambda} + \eta^{\nu}{}_{\lambda}\gamma^{\mu}\partial_{\kappa},\\
&(\underline{6})^{\mu\nu}{}_{\kappa\lambda} = (\partial^{\mu}\gamma^{\nu} + \partial^{\nu}\gamma^{\mu})\eta_{\mu\lambda}, \qquad (\underline{7})^{\mu\nu}{}_{\kappa\lambda} = \eta^{\mu\nu}(\partial_{\kappa}\gamma_{\lambda} + \partial_{\lambda}\gamma_{\kappa}),\\
&(\underline{8})^{\mu\nu}{}_{\kappa\lambda} = \Box^{-1}\not\partial\,\partial^{\mu}\partial^{\nu}\eta_{\kappa\lambda}, \qquad (\underline{9})^{\mu\nu}{}_{\kappa\lambda} = \Box^{-1}\not\partial\,\eta^{\mu\nu}\partial_{\kappa}\partial_{\lambda},\\
&(\underline{10})^{\mu\nu}{}_{\kappa\lambda} = \Box^{-1}\not\partial(\eta^{\mu}{}_{\kappa}\partial^{\nu}\partial_{\lambda} + \eta^{\mu}{}_{\lambda}\partial^{\nu}\partial_{\kappa} + \eta^{\nu}{}_{\kappa}\partial^{\mu}\partial_{\lambda} + \eta^{\nu}{}_{\lambda}\partial^{\mu}\partial_{\kappa}),\\
&(\underline{11})^{\mu\nu}{}_{\kappa\lambda} = \Box^{-1}(\partial^{\mu}\gamma^{\nu} + \partial^{\nu}\gamma^{\mu})\not\partial(\partial_{\kappa}\gamma_{\lambda} + \partial_{\lambda}\gamma_{\kappa}),\\
&(\underline{12})^{\mu\nu}{}_{\kappa\lambda} = \Box^{-1}\partial^{\mu}\partial^{\nu}(\partial_{\kappa}\gamma_{\lambda} + \partial_{\lambda}\gamma_{\kappa}),\\
&(\underline{13})^{\mu\nu}{}_{\kappa\lambda} = \Box^{-1}(\partial^{\mu}\gamma^{\nu} + \partial^{\nu}\gamma^{\mu})\partial_{\kappa}\partial_{\lambda}, \qquad (\underline{14})^{\mu\nu}{}_{\kappa\lambda} = \Box^{-2}\not\partial\,\partial^{\mu}\partial^{\nu}\partial_{\kappa}\partial_{\lambda}.
\end{aligned}
\tag{A3}
$$

Gauge transformations are represented with the help of the vector-bispinor ε^{μ}. As we have shown, for the gauge transformations only the representation $4 = (1, \frac{1}{4}) \oplus (\frac{1}{2}, 1)$ is needed. The following projection operators β^s_{i4} and β^s_{4i} extracted from the vector-bispinor the needed representation 4:

$$
\begin{aligned}
(\sqrt{\Box}\,\beta^{3/2}_{14})^{\mu\nu}{}_{\kappa} = (1/\sqrt{15})[&-\tfrac{1}{3}\eta^{\mu\nu}\not\partial\,\gamma_{\kappa} + \tfrac{1}{3}\eta^{\mu\nu}\partial_{\kappa} - \tfrac{1}{2}(\eta^{\mu}{}_{\kappa}\gamma^{\nu} + \eta^{\nu}{}_{\kappa}\gamma^{\mu})\not\partial\\
&-\tfrac{5}{6}(\partial^{\mu}\gamma^{\nu} + \partial^{\nu}\gamma^{\mu})\gamma_{\kappa} + 3(\eta^{\mu}{}_{\kappa}\partial^{\nu} + \eta^{\nu}{}_{\kappa}\partial^{\mu})\\
&+(4/3\Box)(\partial^{\mu}\gamma^{\nu} + \partial^{\nu}\gamma^{\mu})\not\partial\,\partial_{\kappa} + (2/\Box)\partial^{\mu}\partial^{\nu}\not\partial\,\gamma_{\kappa} - (8/\Box)\partial^{\mu}\partial^{\nu}\partial_{\kappa}],
\end{aligned}
$$

$$(\sqrt{\Box}\beta^{3/2}_{41})^{\kappa}{}_{\rho\sigma}=(1/\sqrt{15})[-\tfrac{1}{3}\gamma^{\kappa}\partial\!\!\!/\,\eta_{\rho\sigma}+\tfrac{1}{3}\partial^{\kappa}\eta_{\rho\sigma}-\tfrac{1}{2}\partial\!\!\!/(\eta^{\kappa}{}_{\rho}\gamma_{\sigma}+\eta^{\kappa}{}_{\sigma}\gamma_{\rho})$$
$$-\tfrac{5}{6}\gamma^{\kappa}(\partial_{\rho}\gamma_{\sigma}+\partial_{\sigma}\gamma_{\rho})+3(\eta^{\kappa}{}_{\rho}\partial_{\sigma}+\eta^{\kappa}{}_{\sigma}\partial_{\rho})$$
$$+(4/3\Box)\partial^{\kappa}\partial\!\!\!/(\partial_{\rho}\gamma_{\sigma}+\partial_{\sigma}\gamma_{\rho})+(2/\Box)\gamma^{\kappa}\partial\!\!\!/\partial_{\rho}\partial_{\sigma}-(8/\Box)\partial^{\kappa}\partial_{\rho}\partial_{\sigma}],$$
$$(\sqrt{\Box}\beta^{1/2}_{14})^{\mu\nu}{}_{\kappa}=(1/6\sqrt{6})(\eta^{\mu\nu}\partial\!\!\!/+\partial^{\mu}\gamma^{\nu}+\partial^{\nu}\gamma^{\mu}-(6/\Box)\partial^{\mu}\partial^{\nu}\partial\!\!\!/)[\gamma_{\kappa}-(4/\Box)\partial\!\!\!/\partial_{\kappa}],$$
$$(\sqrt{\Box}\beta^{1/2}_{41})^{\kappa}{}_{\rho\sigma}=(1/6\sqrt{6})[\gamma^{k}-(4/\Box)\partial\!\!\!/\partial^{\kappa}][\partial\!\!\!/\,\eta_{\rho\sigma}+\partial_{\rho}\gamma_{\sigma}+\partial_{\sigma}\gamma_{\rho}-(6/\Box)\partial\!\!\!/\partial_{\rho}\partial_{\sigma}],$$
$$(\sqrt{\Box}\beta^{3/2}_{24})^{\mu\nu}{}_{\kappa}=(1/6\sqrt{3})[2\eta^{\mu\nu}\partial\!\!\!/\gamma_{\kappa}-2\eta^{\mu\nu}\partial_{\kappa}-(\partial^{\mu}\gamma^{\nu}+\partial^{\nu}\gamma^{\mu})\gamma_{\kappa}$$
$$+3(\eta^{\mu}{}_{\kappa}\gamma^{\nu}+\eta^{\nu}{}_{\kappa}\gamma^{\mu})\partial\!\!\!/-(2/\Box)(\partial^{\mu}\gamma^{\nu}+\partial^{\nu}\gamma^{\mu})\partial\!\!\!/\partial_{\kappa}], \tag{A4}$$
$$(\sqrt{\Box}\beta^{3/2}_{42})^{\kappa}{}_{\rho\sigma}=(1/6\sqrt{3})[2\gamma^{\kappa}\partial\!\!\!/\,\eta_{\rho\sigma}-2\partial^{\kappa}\eta_{\rho\sigma}-\gamma^{\kappa}(\partial_{\rho}\gamma_{\sigma}+\partial_{\sigma}\gamma_{\rho})$$
$$+3\partial\!\!\!/(\eta^{\kappa}{}_{\rho}\gamma_{\sigma}+\eta^{\kappa}{}_{\sigma}\gamma_{\rho})-(2/\Box)\partial^{\kappa}\partial\!\!\!/(\partial_{\rho}\gamma_{\sigma}+\partial_{\sigma}\gamma_{\rho})],$$
$$(\sqrt{\Box}\beta^{1/2}_{24})^{\mu\nu}{}_{\kappa}=(1/6\sqrt{3})(\tfrac{1}{2}\partial\!\!\!/\,\eta^{\mu\nu}-\partial^{\mu}\gamma^{\nu}-\partial^{\nu}\gamma^{\mu})[\gamma_{\kappa}-(4/\Box)\partial\!\!\!/\partial_{\kappa}],$$
$$(\sqrt{\Box}\beta^{1/2}_{42})^{\kappa}{}_{\rho\sigma}=(1/6\sqrt{3})[\gamma^{\kappa}-(4/\Box)\partial\!\!\!/\partial^{\kappa}](\tfrac{1}{2}\partial\!\!\!/\,\eta_{\rho\sigma}-\partial_{\rho}\gamma_{\sigma}-\partial_{\sigma}\gamma_{\rho}),$$
$$(\sqrt{\Box}\beta^{1/2}_{34})^{\mu\nu}{}_{\kappa}=(1/4\sqrt{3})\eta^{\mu\nu}(\partial\!\!\!/\gamma_{\kappa}-4\partial_{\kappa}),$$
$$(\sqrt{\Box}\beta^{1/2}_{43})^{\kappa}{}_{\rho\sigma}=(1/4\sqrt{3})(\gamma^{\kappa}\partial\!\!\!/-4\partial^{\kappa})\eta_{\rho\sigma}.$$

The operators β^{s}_{i4} and β^{s}_{4i} also satisfy the relations (A1).

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